VECTOR BUNDLES WITH NO INTERMEDIATE COHOMOLOGY ON FANO THREEFOLDS OF TYPE V_{22}

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Abstract. We classify rank-2 vector bundles with no intermediate cohomology on the general prime Fano threefold of index 1 and genus 12. The structure of their moduli spaces is given by means of a monad–theoretic resolution in terms of exceptional bundles.

1. Introduction

The study of vector bundles with no intermediate cohomology, also called arithmetically Cohen–Macaulay bundles (see Definition 2.1), has been taken up by several authors. The well–known splitting criterion for the projective spaces showed by Horrocks in [Hor64] has been generalized by Ottaviani in [Ott89] and [Ott87] to Grassmannians and quadrics. Knörrer in [Knö87] proved that line bundles and spinor bundles are the only aCM bundles on quadrics, while Buchweitz Greuel and Schreyer showed in [BGS87] that only projective spaces and quadrics admit a finite number of equivalence classes of aCM bundles up to twist.

On the other hand, the problem of classifying aCM bundles on special classes of varieties has been studied in several papers. Costa and the first author took up the case of prime Fano threefolds of index 2 in [AC00], while the second author in [Fae03] considered the case of the index-2 prime threefold V_5.

On the other hand Madonna classified rank-2 aCM bundles on the quartic threefold in [Mad00]. He also got in [Mad02] a numerical classification of the invariants of these bundles on any prime Fano threefold V_{2g-2} of index 1 and genus g, $2 \leq g \leq 12$, $g \neq 11$. In particular he conjectured that all the cases of this classification occur on any such threefold V_{2g-2}.

For higher dimensional varieties, the case of $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^4)$ has been studied by Graña and the first author in [AG99].

Here we consider rank-2 aCM bundles on the general prime Fano threefold X of index 1 and genus 12 (see Definition 2.3). We write a bundle the Chern classes of a sheaf $\mathscr{F}$ on X as integers (see Section 2), and we denote a rank-2 sheaf $\mathscr{F}$ on X with $c_1(\mathscr{F}) = c_1$ and $c_2(\mathscr{F}) = c_2$ by $\mathscr{F}_{c_1,c_2}$.

The main result of this paper is the following Theorem.

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Theorem. *On the general X there exist the following vector bundles with no intermediate cohomology:*

i) The bundle $\mathscr{F}_{-1,1}$ associated to a line contained in X;
ii) The bundle $\mathscr{F}_{0,2}$ associated to a conic contained in X;
iii) The bundle $\mathscr{F}_{-1,d}(1)$ associated to an elliptic curve of degree d, with $7 \leq d \leq 14$;
iv) The bundle $\mathscr{F}_{0,4}(1)$ associated to a canonical curve of degree 26 and genus 14 contained in X;
v) The bundle $\mathscr{F}_{-1,15}(2)$ associated to a half–canonical curve C^{59}_{60} of degree 59 and genus 60 contained in X.

These are the only possible indecomposable vector bundles with no intermediate cohomology on X up to isomorphism and twist by line bundles.

The moduli space of semistable vector bundles with no intermediate cohomology is generically smooth of dimension equal to 2 in Case (ii), $2d - 14$ in Case (iii), 5 in Case (iv) 16 in Case (v).

This gives a complete classification of aCM rank-2 bundles on the general Fano threefold X together with a description of the moduli space of all of them. The main tools to show the above theorem are the study of elliptic curves in X and the resolution of the diagonal on $X \times X$ obtained in [Fae04].

The paper is structured as follows. In Section 2 we give basic definitions and we review some known facts concerning the threefold X. We will frequently use the available descriptions of X which we recall in Subsections 2.1, 2.2, 2.3 and 2.4 for the reader's convenience.

In Section 3 we consider briefly lines and conics contained in X and we also give a monad–theoretic interpretation of the Hilbert scheme of lines and conics contained in X. In Sections 4 and 5 we take up the analysis of elliptic, canonical and half–canonical curves in X which give rise to vector bundles with no intermediate cohomology, proving their existence and describing the moduli spaces associated to them.

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2. Preliminaries

Let Y be a smooth projective variety, equipped with a very ample line bundle $\mathscr{O}_Y(1)$. Following standard terminology we put the following definition.

Definition 2.1. Given a sheaf $\mathscr{F}$ over Y, we say that $\mathscr{F}$ is *aCM (arithmetically Cohen–Macaulay)* if $\mathrm{H}^p(Y, \mathscr{F}(t)) = 0$, for all $t \in \mathbb{Z}$ and for $0 < p < \dim(Y)$. Equivalently we will say that $\mathscr{F}$ has *no intermediate cohomology*.

Denote the dual of a vector bundle $\mathscr{F}$ by $\mathscr{F}^*$, and recall that if $\mathscr{F}$ has rank 2, we have $\mathscr{F}^* \simeq \mathscr{F}(-c_1(\mathscr{F}))$.

We recall the Hartshorne–Serre correspondence between codimension-2 subvarieties and rank-2 vector bundles, originally introduced in [Ser63], later considered by many authors, see e.g. [Har74], [Vog78], [OSS80].

Definition 2.2. Let Z be a complete subvariety of Y. Then Z is called *subcanonical* if there exists a line bundle $\mathscr{L}$ on Y such that $\mathscr{L}_{|Z} \simeq \omega_Z$.

Let Z be a subcanonical locally complete intersection 2-codimensional subvariety of Y. Then by [OSS80, Theorem 5.1.1] there exist a rank-2 vector bundle $\mathscr{F}_Z$ over Y and a section $s_Z \in \mathrm{H}^0(Y, \mathscr{F}_Z^*)$ such that $Z = \{s_Z = 0\}$ i.e. Z is the zero locus of s_Z. We will say in this case that $\mathscr{F}_Z$ is *associated* to Z. We will denote by $N_{Z,Y}$ the normal bundle of Z in Y and by $J_{Z,Y}$ the ideal sheaf of Z in Y.

In the hypothesis of the above definition we have the fundamental exact sequence:

$$0 \to \det(\mathscr{F}_Z) \to \mathscr{F}_Z \to J_{Z,Y} \to 0 \tag{1}$$

Finally, in these hypothesis we have the adjunction isomorphism:

$$(\mathscr{F}_Z^*)_{|Z} \simeq N_{Z,Y} \tag{2}$$

Definition 2.3. A *prime Fano threefold of index* 1 *and genus* 12 is a 3-dimensional algebraic variety X with $\mathrm{Pic}(X) \simeq \mathbb{Z} = \langle \mathscr{O}_X(1) \rangle$, $\omega_X \simeq \mathscr{O}_X(-1)$, and $\deg(\mathscr{O}_X(1)) = 22$. The last condition is equivalent to the general curve in the linear system of the double linear section of X having genus 12. Any such X is rational, and we have $\mathrm{h}^0(\mathscr{O}_X(1)) = 14$. Further, the i-th Chow group $\mathrm{CH}^i(X)$ is isomorphic to $\mathbb{Z}$ for $i = 1, 2, 3$.

From now on we will denote by X a prime Fano threefold of index 1 and genus 12. We will denote the Chern classes of a sheaf $\mathscr{F}$ on X by integers $c_i \in \mathbb{Z}$ meaning $c_i(\mathscr{F}) = c_i \xi_i$ where ξ_i is the generator of $\mathrm{CH}^i(X) \simeq \mathbb{Z}$ for $i = 1, 2, 3$. Recall that ξ_2 is the class of a line in X. Further, we define $\mu(\mathscr{F})$ as the rational number $c_1(\mathscr{F})/\operatorname{rk}(\mathscr{F})$. We say that a vector bundle $\mathscr{F}$ is normalized if $-\operatorname{rk}(\mathscr{F}) < c_1(\mathscr{F}) \leq 0$, equivalently $\mathscr{F}$ is normalized if $-1 < \mu(\mathscr{F}) \leq 0$. Clearly we have $\mu(\mathscr{F}_1 \otimes \mathscr{F}_2) = \mu(\mathscr{F}_1) + \mu(\mathscr{F}_2)$. We refer to [HL97] for the definition of (semi)stability (in the sense of Mumford and Takemoto). A stable bundle $\mathscr{F}$ with $\mu(\mathscr{F}) < 0$ satisfies $\mathrm{h}^0(\mathscr{F}) = 0$. Recall by Hoppe's criterion that, since $\mathrm{Pic}(X)$ is generated by $\mathscr{O}_X(1)$, a rank-2 bundle $\mathscr{F}$ on X is stable if $\mathrm{h}^0(\mathscr{F}(t)) = 0$, with $c_1(\mathscr{F}(t)) = 0$ or $c_1(\mathscr{F}(t)) = -1$, see e.g. [OSS80, Lemma 1.2.5].

By Hirzebruch–Riemann–Roch's formula for a rank r vector bundle on X with Chern classes c_i we have the formula:

$$\begin{aligned} 6\,\chi(\mathscr{F}(s)) = 22\,s^3\,r + 11\,s^2\,(3\,r + 6\,c_1) + s\,(23\,r + 66\,c_1 - 6\,c_2 + 66\,c_1^2) + \\ + 6\,r + 23\,c_1 - 3\,c_2 - 3\,c_1\,c_2 + 33\,c_1^2 + 3\,c_3 + 22\,c_1^3 \end{aligned}$$

Given a smooth projective variety Y, equipped with a very ample line bundle $\mathscr{O}_Y(1)$, we will write $\mathsf{M}_Y(r; c)$ where r is an integer and c is a string with $c_i \in \mathrm{CH}^i(Y)$ (identified with integers whenever possible) for the moduli space of rank-r semistable vector bundles on Y with Chern classes c_i.

Let P_Z be the Hilbert polynomial of a subscheme $Z \subset Y$ with respect to the polarization $\mathscr{O}_Y(1)$. Given a natural number m, we will denote the Hilbert scheme of subschemes of Y of length m by $\mathrm{Hilb}_m(Y)$. Further, given a reduced curve $Z \subset Y$ of degree d and genus g, we will denote by $\mathcal{H}_{d,g}(Y)$ the irreducible component containing $[Z]$ of the Hilbert scheme of subschemes of Y whose Hilbert polynomial equals P_Z cfr. [HL97, Page 41].

Now, if Z is a reduced curve associated to the rank 2 vector bundle $\mathscr{F}_Z$ over a smooth projective threefold Y, by virtue of the exact sequence (1), the Hilbert polynomial and the Chern classes of $\mathscr{F}_Z$ are determined by the degree d and genus g of Z. Notice also that the bundle $\mathscr{F}_Z$ associated to Z is represented by an element of $\mathrm{Ext}^1_Y(J_{Z,Y}, \det(\mathscr{F}_Z))$.

So, if $\dim \mathrm{Ext}^1_Y(J_{Z,Y}, \det(\mathscr{F}_Z)) = 1$, and if the bundle $\mathscr{F}_Z$ is semistable, the Hartshorne–Serre correspondence provides a rational map:

$$\begin{aligned} \tau : \mathcal{H}_{d,g}(Y) &\dashrightarrow \mathsf{M}_Y(2; c_1(\mathscr{F}_Z), c_2(\mathscr{F}_Z)) \\ [Z] &\mapsto [\mathscr{F}_Z] \end{aligned} \tag{3}$$

In the next subsections we will recall some of the available constructions of the threefold X. We will also sketch the description of four fundamental vector bundles E, U, Q, K, respectively of rank 2, 3, 4, 5, defined over X.

We refer to [Muk92], [Muk03], [Sch01] and [Fae04] for the proofs and some more details.

2.1. **Nets of dual quadrics and twisted cubics.** Let k be an algebraically closed field, $A \simeq k^4$ and $B \simeq k^3$ be k-vector spaces, and let $R(A) = k[A]$ (respectively, $R(B) = k[B]$) be polynomial algebras over A (respectively, over B). Let $\mathrm{S}^d A = R(A)_d$ be the d-th symmetric power of the vector space A. Given a twisted cubic Γ, we have $P_\Gamma(t) = 3t + 1$ and we consider $\mathsf{H} = \mathcal{H}_{3,0}(\mathbb{P}(A))$, as constructed in [EPS87]. Given a twisted cubic $[\Gamma] \in \mathsf{H}$, denote by J_Γ the ideal sheaf of Γ in $\mathbb{P}(A)$. The open subset H_c consisting of points which are Cohen–Macaulay embeds in $\mathbb{G}(k^3, \mathrm{S}^2 A)$ by means of the vector bundle U_{H} whose fiber over $[\Gamma] \in \mathsf{H}_c$ is $\mathrm{Tor}_1^{R[A]}(R[A]/J_\Gamma, k)_2 \simeq k^3$. Equivalently, we associate to any $[\Gamma] \in \mathsf{H}$ the net of quadrics in $\mathbb{P}(A)$ vanishing on Γ.

Definition 2.4. A net of dual quadrics Ψ (parametrized by B) in $\mathbb{P}(A)$ is defined as a surjective map $\Psi : \mathrm{S}^2 A \to B$. Let $V_\Psi = \ker(\Psi)$. Given a general net Ψ we define:

$$\begin{aligned} X_\Psi =& \{[\Gamma] \in \mathsf{H} \subset \mathcal{H}_{3,0}(\mathbb{P}(A)) | \Psi(\mathrm{H}^0(J_\Gamma(2))) = 0\} = \\ =& \{[\Gamma] \in \mathsf{H} \subset \mathcal{H}_{3,0}(\mathbb{P}(A)) | \mathrm{H}^0(J_\Gamma(2)) \subset V_\Psi\} \end{aligned}$$

We define the bundle U on X as the restriction to X of U_{H}.

Definition 2.5. Let Ψ be a general net of dual quadrics and set $X = X_\Psi$. Then there is a rank-2 vector bundle E on X defined by $E_{[\Gamma]} = \mathrm{Tor}_2^{R[A]}(R[A]/J_\Gamma, k)_3 \simeq k^2$. Equivalently we associate to any $[\Gamma] \in \mathsf{H}$ its space of first–order syzygies.

We recall the following lemma from [Fae04, Lemma 6.3].

Lemma 2.6. *The bundle E^* is globally generated and aCM with $\mathrm{h}^0(E^*) = 8$. There is a rank-6 bundle $E' = \ker(\mathrm{H}^0(E^*) \otimes \mathscr{O} \to E^*)$. The bundle E' is also stable and aCM.*

2.2. **Plane quartics.** Let B be a 3-dimensional k-vector space and $F \in \mathrm{S}^4 B$ be a plane quartic. Put $\check{\mathbb{P}}^2 = \mathbb{P}(B^*)$. Consider the Hilbert scheme $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)$ of zero–dimensional length 6 closed subschemes of $\check{\mathbb{P}}^2$. We define the subvariety of $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)$ consisting of polar hexagons to F.

$$X_F = \{\Lambda = (f_1, \dots, f_6) \in \mathrm{Hilb}_6(\check{\mathbb{P}}^2) \mid f_1^4 + \cdots + f_6^4 = F\}$$

Lemma 2.7 (Mukai, Schreyer)**.** *For general F the variety X_F is a prime Fano threefold of index 1 and genus 12. Given a net of dual quadrics Ψ, there exists a quartic form F such that $X_F \simeq X_\Psi$.*

Definition 2.8. Let F be a general plane quartic and let $X = X_F$. Then there is a rank-5 vector bundle K on X_F defined over an element $\Lambda = (f_1, \dots, f_6) \in X_F$ by $K_\Lambda = \langle f_1^4, \dots, f_6^4 \rangle / F$. The bundle K^* is stable and aCM (cfr. [Fae04, Lemma 6.2]) with $\mathrm{h}^0(K^*) = 14$ and $c_1(K) = -2$ (cfr. [Fae04, Lemma 6.1]).

Remark 2.9. Under the hypothesis of Lemma 2.7, there is a natural isomorphism $V_\Psi \simeq \mathrm{S}^3 B / F(B^*)$, where we consider F as a map $B^* \to \mathrm{S}^3 B$ taking an element $\partial \in B^*$ to the cubic form $\partial(F)$ (apolarity action). We set $V_F = \mathrm{S}^3 B / F(B^*)$. The fiber of U over an element $\Lambda = (f_1, \dots, f_6) \in X_F$ is naturally identified with $\langle f_1^3, \dots, f_3^4 \rangle / F(B^*)$. The global sections of U^* (respectively, of K^*) are then identified with $V_F = \mathrm{S}^3 B / F(B^*)$ (respectively, with $\mathrm{S}^4 B / F$). An element ∂ of B^* gives a map $\mathrm{S}^4 B \to \mathrm{S}^3 B$ by apolarity action and therefore a homomorphism $\partial : K \to U$.

2.3. **Nets of alternating 2-forms.** Let V (respectively, B) be a 7-dimensional (respectively, 3-dimensional) k-vector space and let G be the Grassmannian $\mathbb{G}(k^3, V)$. Define U_G (resp. Q_G) as the universal rank 3 subbundle (resp. the universal rank 4 quotient bundle) on G and let σ be a section of $B^* \otimes \wedge^2 U_\mathsf{G}^*$. Equivalently σ is a net of alternating 2-forms i.e. $\sigma \in B^* \otimes \wedge^2 V^*$.

Definition 2.10. Define X_σ as the zero locus in G of $\sigma \in B^* \otimes \wedge^2 V^*$. For general σ the variety X_σ is a prime Fano threefold of index 1 and genus 12.

Lemma 2.11 (Mukai)**.** *Given a general plane quartic F there is a net of alternating 2-forms σ_F such that $X_\sigma \simeq X_F$.*

From now on we identify X with $X_\Psi \simeq X_F \simeq X_\sigma$ where Ψ is a general net of dual quadrics, F is the quartic form provided by Lemma 2.7 and σ is the net of alternating 2-forms given by Lemma 2.11. In particular, we fix a 3-dimensional (respectively, 4-dimensional) k-vector space B (respectively, A). Recall by Remark 2.9 that we have $V \simeq V_F \simeq V_\Psi$. We observe also that in our hypothesis we have $(U_\mathsf{G})_{|X} \simeq (U_\mathsf{H})_{|X}$, so we denote by U also the restriction to X_σ of the vector bundle U_G. We set $Q = (Q_\mathsf{G})_{|X}$.

Lemma 2.12. *There are the following natural isomorphisms:*

$$\mathrm{Hom}(U, Q^*) \simeq B \qquad \mathrm{Hom}(E, U) \simeq A^* \tag{4}$$

$$\mathrm{Hom}(K, U) \simeq B^* \qquad \mathrm{Hom}(E, K) \simeq A \tag{5}$$

Moreover there are the following exact sequences:

$$0 \longrightarrow U \longrightarrow V \otimes \mathscr{O} \longrightarrow Q \longrightarrow 0 \tag{6}$$

$$0 \longrightarrow K \longrightarrow B \otimes U \longrightarrow Q^* \longrightarrow 0 \tag{7}$$

$$0 \longrightarrow \wedge^2 U \longrightarrow A \otimes E \longrightarrow K \longrightarrow 0 \tag{8}$$

$$0 \to E \to \mathscr{O}^{\oplus 8} \to (E')^* \to 0 \tag{9}$$

The Chern classes of these bundles are:

$$\begin{array}{lll}
c_1(E) = -1 & c_2(E) = 7 & \\
c_1(U) = -1 & c_2(U) = 10 & c_3(U) = -2 \\
c_1(Q^*) = -1 & c_2(Q^*) = 12 & c_3(Q^*) = -4 \\
c_1(K) = -2 & c_2(K) = 40 & c_3(K) = -20 \\
c_1(E') = -1 & c_2(E') = 15 & c_3(E') = -8
\end{array}$$

Proof. The exact sequences (6) and (7) are proved in [Fae04, Lemma 6.1], together with (4) and the first isomorphism in (5).

On the other hand, (8) follows by [Fae04, Proposition 6.4] and (9) is Lemma 2.6. The second isomorphism in (5) follows from [Fae04, Corollary 6.8].

The Chern classes of U, Q^* and $\wedge^2 U$ are easily computed by restriction from $\mathbb{G}(k^3, V)$. Finally, the Chern classes of K, E and E' follow from the exact sequences (7), (8) and (9). □

2.4. **Birational Geometry.** We resketch briefly the birational geometry of X following [Isk78], [Isk89]. Fano's double projection from a line is used there and we refer to [IP99] for a complete treatment.

Let V_5 be the del Pezzo threefold obtained cutting $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^4) \subset \mathbb{P}^9$ with a general $\mathbb{P}^6 \subset \mathbb{P}^9$ and denote by S_5 a general hyperplane section of V_5.

It turns out that X is birational to V_5 under the double projection from a line contained in X and we will use this map to embed in X some elliptic curves contained in V_5.

The divisor S_5 is a degree 5 del Pezzo surface, hence isomorphic to the blow–up of $\mathbb{P}^2$ at 4 points $B_1, \ldots, B_4$. Further we have $\omega_{S_5}^* \simeq \mathscr{O}_{S_5}(1) \simeq \mathscr{O}(3\,\ell - \sum b_i)$ where ℓ is the class of a line in $\mathbb{P}^2$ and b_i is the exceptional divisor over the point B_i.

Recall by [IP99] that the threefold V_5 contains a rational normal curve C_0^5 of degree 5 (restrict to S_5 and take the divisor $2\ell - b_1$). Further C_0^5 has exactly 3 chords T_i, $i = 1, 2, 3$. Indeed any chord of C_0^5 is contained in S_5 and the only lines in S_5 meeting C_0^5 at two points are of the form $\ell - b_i - b_j$, for $1 < i < j$.

Denoting by H_{V_5} the divisor associated to $\mathscr{O}_{V_5}(1)$, the linear system $3H_{V_5} - 2C_0^5$ defines a birational map $\varphi : V_5 \dashrightarrow X$. Let $\tilde{X}$ be the variety obtained by blowing up V_5 along C_0^5 and then along the proper preimages of T_j for $j = 1, \ldots, 3$ and ψ_1 the contraction to V_5. There exists a contraction $\tilde{X} \xrightarrow{\psi_2} X$ and we have $\varphi \circ \psi_1 = \psi_2$.

Definition 2.13. Let us fix a general hyperplane section S_5 of V_5 and an isomorphism $S_5 \to \mathrm{Bl}_{B_1,\dots,B_4}(\mathbb{P}^2)$ (there is a finite number of such isomorphisms). Let b_i be the exceptional divisors on S_5 over B_i. For a given rational normal curve $C_0^5 \subset V_5$ with chords $\{T_i, i = 1,2,3\}$ let $\{e_1,\dots,e_5\} = S_5 \cap C_0^5$ and $f_i = S_5 \cap T_i$. On S_5 we define $\mathscr{L} = 9\ell - 3\sum b_i - 2\sum e_j - \sum f_k$ and we have $\varphi_{|S_5} = \varphi_{|\mathscr{L}|}$, where $\varphi_{|\mathscr{L}|}$ is the map associated to the linear system $|\mathscr{L}|$.

2.5. **Resolution of the diagonal.** We will recall here the resolution of the diagonal on X and the induced Beilinson theorem. We refer to [Gor90], [Rud90], [Dre86] for general setup on exceptional collection and mutations.

Let us define the collection $(G_3,\dots,G_0) = (E, U, Q^*, \mathscr{O})$. This collection is strongly exceptional i.e. $\mathrm{Ext}^p(G_j, G_i) = 0$ if $p > 0$ or if $i > j$. This is proved in [Kuz96]. Furthermore we define the collection $(G^3,\dots,G^0) = (E, K, U, \mathscr{O})$. The following lemma, which is proved in [Fae04, Theorem 7.2], states that these two collections fit together to give a resolution of $\mathscr{O}_\Delta$ over $X \times X$.

Lemma 2.14. *For general X there exists a resolution of $\mathscr{O}_\Delta$ on $X \times X$ of the form:*

$$0 \to G_3 \boxtimes G^3 \to \cdots \to G_0 \boxtimes G^0 \to \mathscr{O}_\Delta \to 0$$

Any coherent sheaf $\mathscr{F}$ on X is functorially isomorphic to the cohomology of a complex $\mathcal{C}_{\mathscr{F}}$ whose terms are given by:

$$\mathcal{C}^k_{\mathscr{F}} = \bigoplus_{i-j=k} \mathrm{H}^i(\mathscr{F} \otimes G^j) \otimes G_j$$

Alternatively $\mathscr{F}$ is functorially isomorphic to the cohomology of a complex $\mathcal{D}_{\mathscr{F}}$ whose terms are given by:

$$\mathcal{D}^k_{\mathscr{F}} = \bigoplus_{i-j=k} \mathrm{H}^i(\mathscr{F} \otimes G_j) \otimes G^j$$

We have the following consequence of Lemma 2.14, namely Castelnuovo–Mumford regularity associated to the collection $(G_3,\dots,G_0)$, cfr. [Fae04, Corollary 7.4].

Corollary 2.15. *Let $\mathscr{F}$ be a coherent sheaf on X and suppose $\mathrm{H}^p(G_p \otimes \mathscr{F}) = 0$ for $p > 0$. Then $\mathscr{F}$ is globally generated.*

2.6. **Vector bundles with no intermediate cohomology.** Recall from the introduction that a rank 2 vector bundle $\mathscr{F}$ with $c_1(\mathscr{F}) = c_1$ and $c_2(\mathscr{F}) = c_2$ is denoted by $\mathscr{F}_{c_1,c_2}$. Similarly, a curve of genus g and degree d is denoted by C_g^d.

Lemma 2.16 (Madonna)**.** *The only possible classes of indecomposable normalized rank-2 aCM vector bundles on X up to isomorphism are the following:*

i) The unstable bundle $\mathscr{F}_{-1,1}$ associated to a line in X;
ii) The semistable bundle $\mathscr{F}_{0,2}$ associated to a conic in X;
iii) The stable bundle $\mathscr{F}_{-1,d}(1)$ associated to an elliptic curve C_1^d contained in X with $7 \le d \le 14$;

iv) The stable bundle $\mathscr{F}_{0,4}(1)$ *associated to a canonical curve* C^{26}_{14} *contained in* X*;*

v) The stable bundle $\mathscr{F}_{-1,15}(2)$ *associated to a half–canonical curve* C^{59}_{60} *contained in* X*.*

In any of these cases, the smallest $t \in \mathbb{Z}$ *such that* $\mathrm{h}^0(\mathscr{F}(t)) \neq 0$ *is the one stated.*

Proof. We refer to [Mad02] for the full proof, with the only exception of condition $d \geq 7$ in (iii) which we show at the end of Section 4. However, we sketch here the main argument used in [Mad02]. Considering the first twist $\mathscr{F}_{c_1,c_2}$ of $\mathscr{F}$ with a nonzero global section s, one proves easily that $Z = \{s = 0\}$ is a connected curve of arithmetic genus $1 + 1/2(c_1c_2 - c_2)$ and degree c_2. Therefore $c_1 \geq 1 - 2/c_2 \geq -1$, so $\mathscr{F}$ is stable except for $c_1 = -1, 0$, which correspond respectively to Cases (i) and (ii).

For $c_1 = 1$ we end up in Case (iii) and, making use of (1), it is easy to check that $d \leq 14$.

For $c_1 > 1$ we find $\mathrm{h}^p(\mathscr{F}_{c_1,c_2}(-1)) = 0$ and $\mathrm{h}^p(\mathscr{F}_{c_1,c_2}(-2)) = 0$ for any p. Take now the following polynomial equations in the variables c_1 and c_2

$$\begin{cases} \chi(\mathscr{F}_{c_1,c_2}(-1)) = 0 \\ \chi(\mathscr{F}_{c_1,c_2}(-2)) = 0 \end{cases}$$

When $c_1 > 1$ we find as only solutions Cases (iv) and (v). $\square$

3. Lines and conics

It is classically known that X contains a one–dimensional family of lines and a two–dimensional family of smooth conics (see [IP99, Propositions 4.2.2 and 4.2.5] and references therein). Denote a line (respectively, a conic) in X by C^1_0 (respectively, by C^2_0). Here we will just provide resolutions of the sheaf $\mathscr{O}_{C^1_0}(-1)$ and of the bundle $\mathscr{F}_{C^2_0}$ with respect to the collection $(G_3, \ldots, G_0)$. This will give a straightforward description of the Hilbert schemes of lines and conics in X.

Lemma 3.1. *The sheaf* $\mathscr{O}_{C^1_0}(-1)$ *admits the resolution:*

$$0 \to E \to K \xrightarrow{\alpha_{C^1_0}} U \to \mathscr{O}_{C^1_0}(-1) \to 0 \tag{10}$$

The map $\alpha_{C^1_0} \in \mathrm{Hom}(K,U) \simeq B^*$ *degenerates along a line* C^1_0 *if and only if it lies in the discriminant quartic curve* $\det(\Psi^\top) \subset \check{\mathbb{P}}^2 = \mathbb{P}(B^*)$*. In particular the Hilbert scheme of lines in* X *is isomorphic to the curve* $\det(\Psi^\top)$*.*

Proof. Clearly we have $(G_j)_{C^1_0} \simeq \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}^{4-j}$. It follows that $\mathrm{h}^1(G_j \otimes \mathscr{O}_{C^1_0}(-1)) = 1$ for $j = 3, 2, 1$, so by Lemma 2.14 the sheaf $\mathscr{O}_{C^1_0}(-1)$ admits the resolution (10).

The Hilbert scheme of lines in X is isomorphic to the curve $\det(\Psi^\top)$ by [Sch01, Theorem 6.1]. However here we sketch a simpler argument. Recall by (5) the isomorphism $\mathrm{Hom}(K,U) \simeq B^*$. Applying the functor $\mathrm{Hom}(E,-)$ to a morphism $\alpha : K \to U$, corresponds to the linear map $\alpha \mapsto \Psi^\top(\alpha)$ under the morphism $\Psi^\top : B^* \to \mathrm{S}^2 A^*$ i.e. α is taken by $\mathrm{Hom}(E,-)$ to a linear map $A \xrightarrow{\Psi^\top(\alpha)} A^*$. Since $\mathrm{Hom}(E, K) \otimes E \to K$ and $\mathrm{Hom}(E, U) \otimes E \to U$ are epimorphisms, it follows that $\mathrm{Hom}(E, \alpha)$ is surjective if and only if α is surjective. This fails to hold precisely if α lies in the discriminant curve $\det(\Psi^\top)$, in which case there is a unique map $E \to \ker(\alpha)$. This map is an isomorphism and we see that $\mathrm{coker}(\alpha)$ is isomorphic to $\mathscr{O}_{C_0^1}(-1)$ by a Hilbert polynomial computation. $\square$

Lemma 3.2 (Takeuchi)**.** *Through any point in X there exists a finite number of conics contained in X. The Hilbert scheme of conics in X is isomorphic to $\mathbb{P}(B)$.*

Proof. The first statement is proved in [Tak89]. Also one may consult [IP99, Lemma 4.2.6]. For any conic C_0^2 in X there exists an exact sequence:

$$0 \to U \to Q^* \to J_{C_0^2,X} \to 0 \tag{11}$$

On the other hand any homomorphism $U \to Q^*$ degenerates along a conic. Since $\mathrm{Hom}(U, Q^*) \simeq B$ the lemma is proved. $\square$

The previous lemma allows us to formulate the following corollary.

Corollary 3.3. *The set of stable points in moduli space $\mathsf{M}_X(2; 0, 2)$ is empty. The set of semistable points is isomorphic to $\mathbb{P}^2 = \mathbb{P}(B)$. The bundle $\mathscr{F}_{0,2}$ of Lemma 2.16, Case (ii) admits the following resolution:*

$$0 \to U \to Q^* \oplus \mathscr{O} \to \mathscr{F}_{0,2} \to 0$$

Proof. Since the bundle $\mathscr{F}_{0,2}$ admits a unique global section s, and since s vanishes along a conic C_0^2 there is an isomorphism between $\mathsf{M}_X(2; 0, 2)$ and $\mathcal{H}_{2,0}(X) \simeq \mathbb{P}^2$ (the Hilbert scheme of conics contained in X). The bundle $\mathscr{F}_{0,2}$ is strictly semistable for $c_1(\mathscr{F}) = 0$.

The exact sequence (1) in this case reads:

$$0 \to \mathscr{O} \to \mathscr{F}_{0,2} \to J_{C_0^2,X} \to 0 \tag{12}$$

Since $\mathrm{Ext}^1(Q^*, \mathscr{O}) = 0$, any morphism $Q^* \to J_{C_0^2,X}$ lifts to a morphism $Q^* \to \mathscr{F}_{0,2}$. Considering the map $\mathscr{O} \to \mathscr{F}_{0,2}$ in the exact sequence (12) and lifting the projection $Q^* \to J_{C_0^2,X}$ in the exact sequence (11) we obtain a surjective bundle map $Q^* \oplus \mathscr{O} \to \mathscr{F}_{0,2}$ whose kernel is isomorphic to U. This provides the desired resolution. $\square$

4. Elliptic curves

In this section we prove the existence of elliptic curves in X with the properties required by Case (iii) of Lemma 2.16. In particular the degree of these curves varies from 7 to 14 and we deal with the case $7 \le d \le 13$ in Proposition 4.1. Case $d = 14$ is considered in Proposition 4.4 where we consider also $d = 15$ which we will need in Section 5.

Proposition 4.1. *On the general variety X there exist smooth elliptic curves C_1^d of degree d for $7 \le d \le 13$. The curve C_1^d is contained in exactly $14 - d$ independent hyperplanes.*

We will construct smooth elliptic curves in X by means of the birational map $\varphi : V = V_5 \dashrightarrow X$ described in Subsection 2.4.

Lemma 4.2. *Let $S = S_5$ be a fixed hyperplane section of V and fix notation as in Subsection 2.4. The irreducible component $\mathcal{H}_{5,0}(V)$ of the Hilbert scheme containing smooth rational normal quintics in V has dimension 10 at general $[C_0^5]$ and there is a dominant rational map $\zeta : \mathcal{H}_{5,0}(V) \dashrightarrow \mathrm{Hilb}_5(\mathbb{P}^2)$ defined by $\zeta : [C_0^5] \mapsto e_1 + \cdots + e_5$.*

Proof. Set $C = C_0^5$. First notice that by the Riemann–Roch formula we have $\mathrm{expdim}(\mathscr{T}_{\mathcal{H}_{5,0}(V),[C]}) = 10$ because $\deg(N_{C,V}) = 10$ so that $\chi(N_{C,V}) = 10$.

Since $C \subset S$, we have the exact sequence of normal bundles:

$$0 \to N_{C,S} \to N_{C,V} \to (N_{S,V})_{|C} \to 0$$

Now, computing $(2\ell - b_1)^2 = 3$, after the identification $C \simeq \mathbb{P}^1$ we get $N_{C,S} \simeq \mathscr{O}_{\mathbb{P}^1}(3)$ and we obtain an exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^1}(3) \to N_{C,V} \to \mathscr{O}_{\mathbb{P}^1}(5) \to 0$$

Therefore $\mathrm{h}^0(N_{C,V}) = \chi(N_{C,V}) = 10$ so $\mathcal{H}_{5,0}(V)$ is smooth and 10-dimensional.

Let $\mathbb{P}(\mathrm{H}^0(V, \mathscr{O}_V(1))) = \mathbb{P}^6$. Notice that, once we fix the hyperplane section S, for any curve C, the intersection $C \cap S$ gives 5 points spanning $\mathbb{P}^4 \subset \mathbb{P}^6$. Conversely, given any $\mathbb{P}^4 \subset \mathbb{P}^6$, there is a curve C such that the spaces $\langle C\rangle$, $\langle S\rangle$ span $\mathbb{P}^6$. Fixing S thus provides a birational map $\mathcal{H}_{5,0}(V) \dashrightarrow \mathbb{G}(\mathbb{P}^4, \mathbb{P}^6)$.

Since $\dim(\mathcal{H}_{5,0}(V)) = \dim(\mathrm{Hilb}_5(\mathbb{P}^2)) = 10$, we have to prove that the map ζ is generically finite. So we fix $\underline{e} = (e_1, \dots, e_5)$ and we consider the space $\mathbb{P}^4_{\underline{e}} = \langle e_1, \dots, e_5\rangle$. Varying a hyperplane section S' of V in the pencil of hyperplanes containing $\mathbb{P}^4_{\underline{e}}$, we obtain a ruled surface $S^j_{\underline{e}}$ consisting of exceptional lines in S' of type b'_j. The ruled surface $S^j_{\underline{e}}$ is not a cone for there are finitely many lines through any point in V (see [IP99, Page 64], [FN89]). Thus its dual variety is a hypersurface in $\check{\mathbb{P}}^6$.

Now given a curve $C \subset S'$, we let $C = 2\ell - b'_1$. So we have $\zeta(C) = e_1 + \cdots + e_5$ if and only if there is a hyperplane section $S' = \mathbb{P}^5 \cap V$ with $\mathbb{P}^5 \supset \mathbb{P}^4_{\underline{e}}$ and such that $\mathbb{P}^5$ contains the curve of class $2\ell - b_1$. This happens if and only if the hyperplane $\mathbb{P}^5$ is tangent to the ruled surface $S^1_{\underline{e}}$. Being the dual variety of $S^1_{\underline{e}}$ a hypersurface, it intersects the general pencil of $\mathbb{P}^5$'s containing $\mathbb{P}^4_{\underline{b}}$ in a finite set of points. ☐

Lemma 4.3. *Let S be a fixed hyperplane section and fix notation as in Definition 2.13. Define the following linear systems:*

$$\mathscr{L}_9 = 4\ell - 2b_1 - 2b_2 - b_3 - b_4 - e_1 - e_2 - e_3 - \sum f_j \tag{13}$$

$$\mathscr{L}_{10} = 5\ell - 2\sum b_i - 2e_1 - e_2 - e_3 - \sum f_j \tag{14}$$

$$\mathscr{L}_{11} = 4\ell - 2b_1 - 2b_2 - b_3 - b_4 - e_1 - e_2 - \sum f_j \tag{15}$$

$$\mathscr{L}_{12} = 5\ell - 2\sum b_i - 2e_1 - e_2 - \sum f_j \tag{16}$$

$$\mathscr{L}_{13} = 4\ell - 2b_1 - 2b_2 - b_3 - b_4 - e_1 - \sum f_j \tag{17}$$

Then each $\mathscr{L}_d$ has positive dimension and contains a smooth element $\tilde{C}^d_1$. The curve $\varphi(\tilde{C}^d_1)$ is a smooth elliptic curve in X of degree d contained in precisely $14 - d$ independent hyperplanes.

Proof. The linear systems $\mathscr{L}_j$ just defined have positive dimension by counting parameters, indeed it suffices to compute the expected dimension of the linear system of curves in $\mathbb{P}^2$ with prescribed nodes and passing through assigned points.

For odd (resp., even) d, $\mathscr{L}_d$ contains a smooth element $\tilde{C}_1^d$ if and only if there exists an irreducible plane quartic with nodes only at B_1 and B_2 (resp., an irreducible plane quintic with nodes only at $B_1, \dots, B_6$ and the point in $\mathbb{P}^2$ corresponding to e_1). It suffices to project an elliptic normal quartic (resp., quintic) in $\mathbb{P}^3$ (resp., $\mathbb{P}^4$) from a general point (resp., line) to obtain such a curve.

The degree of $\varphi(\tilde{C}_1^d)$ is easily computed as $d = \mathscr{L}_d \cdot \mathscr{L}$ where $\mathscr{L}$ is the linear system of Definition 2.13.

Since any elliptic curve of degree $d \leq 13$ is contained in a hyperplane section S_{22} of X, we have that $\mathrm{h}^0(J_{C_1^d,X}(1)) = \mathrm{h}^0(J_{C_1^d,S_{22}}(1)) + 1$. Using the map φ and the fixed isomorphism $S \to \mathrm{Bl}_{B_1,\dots B_4}(\mathbb{P}^2)$ we have $\mathrm{h}^0(J_{C_1^d,S_{22}}(1)) = \mathrm{h}^0(\mathbb{P}^2, \mathscr{L} - \mathscr{L}_d)$. We are then reduced to compute the dimension of the following linear systems on $\mathbb{P}^2$:

$$\mathscr{L} - \mathscr{L}_9 = 5\ell - b_1 - b_2 - 2b_3 - 2b_4 - e_1 - e_2 - e_3 - 2e_4 - 2e_5 \tag{18}$$

$$\mathscr{L} - \mathscr{L}_{10} = 4\ell - \sum b_i - e_2 - e_3 - 2e_4 - 2e_5 \tag{19}$$

$$\mathscr{L} - \mathscr{L}_{11} = 5\ell - b_1 - b_2 - 2b_3 - 2b_4 - e_1 - e_2 - 2e_3 - 2e_4 - 2e_5 \tag{20}$$

$$\mathscr{L} - \mathscr{L}_{12} = 4\ell - \sum b_i - e_2 - 2e_3 - 2e_4 - 2e_5 \tag{21}$$

$$\mathscr{L} - \mathscr{L}_{13} = 5\ell - b_1 - b_2 - 2b_3 - 2b_4 - e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5 \tag{22}$$

By Lemma 4.2, we can can compute the dimension of these linear systems choosing the points corresponding to the e_i's in a Zariski open set of $\mathrm{Hilb}_5(\mathbb{P}^2)$. Notice that $\mathrm{expdim}(\mathscr{L} - \mathscr{L}_d) = 13 - d$, so we only need to check that $\mathrm{expdim}(\mathscr{L} - \mathscr{L}_d) = \dim(\mathscr{L} - \mathscr{L}_d)$.

This we can do using Cremona transformations on $\mathbb{P}^2$. For (18) consider the Cremona transformation γ_9 associated the linear system $2\ell - b_3 - b_4 - e_4$. Any curve in $\mathscr{L} - \mathscr{L}_9$ touches a conic through $b_3 - b_4 - e_4$ in 4 points. Further, any curve in $\mathscr{L} - \mathscr{L}_9$ touches the line $\langle B_3, B_4 \rangle$ (resp., $\langle B_4, e_4 \rangle$, $\langle B_3, e_4 \rangle$) in a single further point e_4' (resp., b_3', b_4') so under γ_9 the linear system $\mathscr{L} - \mathscr{L}_9$ is mapped to $4\ell - b_1 - b_2 - b_3' - b_4' - e_1 - e_2 - e_3 - e_4' - 2e_5$. Now the points $e_1, \dots, e_5$ lie in general position by Lemma 4.2 while the points b_i can be chosen generically for we can define S to be the blow–up of $\mathbb{P}^2$ at a general 4-tuple of points.

Since we now have the linear system of plane quartics with only one node and passing through 8 general points, we conclude $\mathrm{h}^0(\mathbb{P}^2, \mathscr{L} - \mathscr{L}_9) = 4$.

In Case (20), γ_{11} is defined as the Cremona transformation associated to $2\ell - b_3 - b_4 - e_3$, sending $\mathscr{L} - \mathscr{L}_9$ to $4\ell - b_1 - b_2 - b_3' - b_4' - e_1 - e_2 - e_3' - 2e_4 - 2e_5$. Now take $\gamma_{11} = \gamma_{2\ell - e_3 - e_4 - b_1}$. Then $\gamma_{11}' \circ \gamma_{11}$ sends $\mathscr{L} - \mathscr{L}_9$ to $3\ell - b_2 - b_3' - b_4' - e_1 - e_2 - e_3' - e_4'' - e_5''$. Now 8 general points impose 8 linearly independent conditions on the 10-dimensional space of plane cubics.

In Case (22) we put $\gamma_{13} = \gamma_{2\ell - b_3 - b_4 - e_2}$ and $\gamma_{13}' = \gamma_{2\ell - e_3 - e_4 - e_5}$. The linear system $\mathscr{L} - \mathscr{L}_{13}$ is mapped by $\gamma_{13}' \circ \gamma_{13}$ to $2\ell - b_2 - b_2 - b_3' - b_4' - e_1 - e_2'$. Since there is no conic through 6 general points we are done.

In Case (19) put $\gamma_{10} = \gamma_{2\ell-e_3-e_4-e_5}$. The lines $\langle e_3, e_4\rangle$ and $\langle e_3, e_5\rangle$ give rise to two extra points e_4' and e_5', so we have to compute $\mathrm{h}^0(3\ell - \sum b_i - e_2 - e_4' - e_5') = 3$.

In Case (21) put $\gamma_{12} = \gamma_{2\ell-e_3-e_4-e_5}$. Here we have no extra points and the statement follows since $\mathrm{h}^0(2\ell - \sum b_i - e_2) = 1$. $\square$

Proof of 4.1. The curve C_1^7 exists according to [Kuz96] and [Fae04], in fact it is just the zero locus of a general global section $s \in \mathrm{H}^0(E^*) \simeq k^8$. For C_1^8, consider a homomorphism $\alpha : K \to U$, where $\alpha \in \mathrm{Hom}(K, U) \simeq B^*$. This morphism is surjective whenever α lies outside the discriminant curve $\det(\Psi^\top) \subset \mathbb{P}(B^*)$ (cfr. Lemma 3.1), so for general α we get a rank-2 locally free sheaf $F_8 = \ker(\alpha)$. It follows easily from Lemma 2.12 that $c_1(F_8) = -1$ and $c_2(F_8) = 8$. Taking global sections of F_8^* and using the identifications of Lemma 2.7 we get:

$$\mathrm{H}^0(F_8^*) \simeq \ker\left(\alpha : \mathrm{S}^4 B/F \to \mathrm{S}^3 B/F(B^*)\right)$$

For general α this map is surjective so $\mathrm{h}^0(F_8^*) = 7$ and F_8^* is globally generated since K^* is. Therefore a general section of F_8^* vanishes along the required curve C_1^8.

For $9 \leq d \leq 13$ the statement follows from Lemma 4.3. $\square$

Proposition 4.4. *On the general variety X there exists a smooth elliptic curve C_1^d of degree d for $d = 14, 15$. In both cases C_1^d is non degenerate.*

Proof. It is well–known that there exist smooth elliptic normal curves of degree 7 in V. However we sketch a quick proof. Denoting by U_V (resp. Q_V), the universal rank-2 subbundle (resp., the universal rank-3 quotient bundle) on $\mathbb{G}(k^2, k^5)$ restricted to V, one proves that for a general map $\alpha : U_V^{\oplus 2} \to (Q_V^*)^{\oplus 2}$, the sheaf $\mathrm{coker}(\alpha) \otimes \mathscr{O}_V(1)$ is a globally generated rank-2 bundle on V whose general section vanishes on the required curve D_7.

Take now a hyperplane section S, denote by $d_1, \ldots d_7$ the intersection points of D_7 with S and recall the notation from Definition 2.13.

Choose a smooth curve C_0^5 in the linear system $2\ell - b_1 - d_1 - d_2 - d_3$. Clearly this linear system has positive dimension. The curve D_7 is mapped by $\varphi_{|\mathscr{L}|}$ to a smooth elliptic curve of degree 15 for it intersects C_0^5 at 3 points with normal crossing. This curve is non degenerate since D_7 is non degenerate too.

Moving the hyperplane section S in $\check{\mathbb{P}}^6$ we can suppose that the point d_4 coincides with the point f_1. Taking again $C_0^5 \in |2\ell - b_1 - d_1 - d_2 - d_3|$ we have that D_7 is now mapped by $\varphi_{|\mathscr{L}|}$ to a non degenerate smooth elliptic curve of degree 14, indeed it intersects C_0^5 (resp. T_1) at 3 points (resp. 1 point) with normal crossing. $\square$

Proposition 4.5. *Let $7 \leq d \leq 15$ and let F_d be the rank-2 vector bundle over X associated to the elliptic curve C_1^d constructed as above. We have $c_1(F_d) = -1$ and $c_2(F_d) = d$. F_d is stable for any d and aCM for $7 \leq d \leq 14$. Moreover we have $\mathrm{h}^0(F_{15}^*) = \mathrm{h}^1(F_{15}^*) = 1$.*

Proof. Set $C = C_1^d$. The numerical invariants of the bundle F_d are obvious and stability follows at once by Hoppe's criterion.

By Serre duality one has $\mathrm{h}^2(F_d^*) = \mathrm{h}^1(F_d(-1)) = \mathrm{h}^1(F_d^*(-2)) = 0$ by (1).

Taking twisted sections in sequence (1) we get that F_d is aCM if and only if $\mathrm{h}^1(F_d(1)) = 0$ i.e. if and only if $\mathrm{h}^1(J_{C,X}(1)) = 0$. Indeed in this case the map $\mathrm{H}^0(\mathscr{O}_X(1)) \to \mathrm{H}^0(\mathscr{O}_C(1))$ is surjective. This implies that $\mathrm{H}^0(\mathscr{O}_X(t)) \to \mathrm{H}^0(\mathscr{O}_C(t))$ is surjective for all $t \geq 1$, so $\mathrm{h}^1(J_{C,X}(t)) = 0$ for $t \geq 1$ so by (1) we get $\mathrm{h}^1(F_d(t)) = 0$ for $t \geq 1$. For $t \leq 0$ this trivially holds too, so F_d is aCM by Serre duality.

This happens precisely when $\mathrm{h}^0(J_{C,X}(1)) = 14 - d$, so the conclusion follows from Propositions 4.1 and 4.4. □

Theorem 4.6. *Let $8 \leq d \leq 15$. Then the bundle F_d of Proposition 4.5 is isomorphic to the cohomology of a monad:*

$$E^{\oplus d-8} \xrightarrow{\beta_d} K^{\oplus d-7} \xrightarrow{\alpha_d} U^{\oplus d-7} \tag{23}$$

For $d = 7$ the bundle F_7 is isomorphic to E.

Proof. By Hirzebruch–Riemann–Roch we get the following equalities:

$$\chi(Q^* \otimes F_d) = d - 7 \tag{24}$$

$$\chi(U \otimes F_d) = d - 7 \tag{25}$$

$$\chi(E \otimes F_d) = d - 8 \tag{26}$$

Now recall that the vector bundles U, Q^*, E and F_d are stable so by [Mar81, Theorem 1.14] any tensor product between them is also a stable vector bundle. This implies at once the following vanishing results:

$$\mathrm{h}^0(Q^* \otimes F_d) = 0$$
$$\mathrm{h}^0(U \otimes F_d) = 0$$
$$\mathrm{h}^0(E \otimes F_d) = 0$$

Serre duality implies the following additional vanishing results:

$$\mathrm{h}^3(Q^* \otimes F_d) = \mathrm{h}^0(Q \otimes F_d) = 0 \qquad \text{because } \mu(Q \otimes F_d) = -1/4 \tag{27}$$

$$\mathrm{h}^3(U \otimes F_d) = \mathrm{h}^0(U^* \otimes F_d) = 0 \qquad \text{because } \mu(U^* \otimes F_d) = -1/6 \tag{28}$$

$$\mathrm{h}^3(E \otimes F_d) = \mathrm{h}^0(E^* \otimes F_d) = 0 \qquad \text{because } c_2(E) \neq c_2(F_d) \tag{29}$$

where (29) follows, since $\mu(E) = \mu(F_d) = -1/2$, but $c_2(E) = 7 \neq d = c_2(F_d)$, so $\mathrm{Hom}(E, F_d) = 0$. Now consider the tensor product of the bundle F_d by the sequences (6) and (9), and by the dual of the sequence (6). Since $\mathrm{h}^0(F_d) = 0$ and $\mathrm{h}^1(F_d) = 0$ we have the equalities:

$$\mathrm{h}^1(Q^* \otimes F_d) = \mathrm{h}^0(U^* \otimes F_d) = 0 \qquad \text{by (28)}$$
$$\mathrm{h}^1(U \otimes F_d) = \mathrm{h}^0(Q \otimes F_d) = 0 \qquad \text{by (27)}$$
$$\mathrm{h}^1(E \otimes F_d) = \mathrm{h}^0((E')^* \otimes F_d)$$

The group $\mathrm{H}^0((E')^* \otimes F_d)$ vanishes as well because E' is also a stable bundle and we have $\mu((E')^* \otimes F_d) = -1/3$. Summing up we have computed:

$$\mathrm{h}^2(Q^* \otimes F_d) = d - 7$$
$$\mathrm{h}^2(U \otimes F_d) = d - 7$$
$$\mathrm{h}^2(E \otimes F_d) = d - 8$$

This implies that F_d is isomorphic to the cohomology of a monad of the form (23). Clearly for $d = 7$ the above argument implies $E \simeq F_7$. $\square$

Theorem 4.7. *Let $7 \le d \le 15$ and let X be general. Then the Hilbert scheme $\mathcal{H}_{d,1}(X)$ of curves in X of degree d and arithmetic genus 1 is smooth of dimension d at a generic point. The moduli space $\mathsf{M}_X(2;-1,d)$ is smooth of dimension $2d-14$ at a generic point.*

Proof. Let $Z = C_1^d$ be a curve of degree d and arithmetic genus 1 contained in X, and consider the vector bundle F_d associated to Z.

Tensoring by F_d the exact sequence (1) and exact sequence defining $Z \subset X$, after the isomorphism (2), we get the following exact sequences:

$$0 \to F_d \to \mathscr{E}nd(F_d) \to F_d^* \otimes J_{Z,X} \to 0 \tag{30}$$

$$0 \to F_d^* \otimes J_{Z,X} \to F_d^* \to N_{Z,X} \to 0 \tag{31}$$

Taking global sections we get $\mathrm{h}^2(X, \mathscr{E}nd(F_d)) = \mathrm{h}^1(Z, N_{Z,X}))$. This means that $\mathsf{M}_X(2;-1,d)$ is unobstructed at $[F_d]$ if and only if $\mathcal{H}_{d,1}(X)$ is unobstructed at $[Z]$.

Consider now the monad (23) given by Theorem 4.6 and denote by W_d^1 (resp., W_d^2) the vector space $\mathrm{H}^2(Q^* \otimes F_d) \simeq k^{d-7}$ (resp., $\mathrm{H}^2(U \otimes F_d) \simeq k^{d-7}$). An element (m,n) of the group $\mathsf{SL}(W_d^1) \times \mathsf{SL}(W_d^2)$ acts on the space $\mathbb{P}(\mathrm{Hom}(K,U) \otimes \mathrm{Hom}(W_d^1, W_d^2))$ taking α_d to $n \circ \alpha_d \circ m^{-1}$. This action is free for general α_d. Taking now the functor $\mathrm{Hom}(E, -)$ we get a morphism:

$$\mathrm{Hom}(K,U) \otimes \mathrm{Hom}(W_d^1, W_d^2) \to A^* \otimes A^* \otimes \mathrm{Hom}(W_d^1, W_d^2) \tag{32}$$

Recall now from (5) that $\mathrm{Hom}(K,U) \simeq B^*$. Hence an element α_d in the vector space $\mathrm{Hom}(K,U) \otimes \mathrm{Hom}(W_d^1, W_d^2)$ can be seen as a map $W_d^1 \to W_d^2$ with entries in B^*. The morphism (32) takes the map α_d to a $4(d-7) \times 4(d-7)$ square matrix $W_d^1 \otimes A \to W_d^2 \otimes A^*$ whose entries are given by $\Psi^\top \otimes \mathrm{id}_{(W_d^1)^*} \otimes \mathrm{id}_{W_d^2}$. Denote this matrix by $\Psi^\top(\alpha_d)$ (see Lemma 3.1).

Consider the sheaf $\ker(\alpha_d : W_d^1 \otimes K \to W_d^2 \otimes U)$. The above discussion implies that there exists an injective map $\beta_d : E^{d-8} \hookrightarrow \ker(\alpha_d)$ if and only if $\mathrm{rk}(\Psi^\top(\alpha_d)) \le 4(d-7) - (d-8) = 3d-20$. Being F_d stable, there is a unique β_d up to isomorphism since $\mathrm{h}^2(E \otimes F_d) = d-8$.

Summing up, there exists an open neighbourhood at $[F_d]$ of an irreducible component of the moduli space $\mathsf{M}_X(2;-1,d)$ which is isomorphic to the set:

$$\mathsf{M}(d) = \{[\alpha_d] \in \mathbb{P}(B^* \otimes \mathrm{Hom}(W_d^1, W_d^2)) \mid \\ \mid \mathrm{rk}(\Psi^\top(\alpha_d)) = 3d-20\} / \mathsf{SL}(d-7) \times \mathsf{SL}(d-7)$$

For sufficiently general $\Psi^\top : B^* \to A^* \otimes A^*$ the variety $\mathsf{M}(d)$ admits smooth points, indeed it is obtained cutting the smooth subset of the variety of $(3d-20)$-secant $(3d-19)$-spaces to the Segre of $\mathbb{P}^{4d-27} \times \mathbb{P}^{4d-27}$ with a sufficiently general linear space.

It is easy to check that the dimension of $\mathsf{M}(d)$ at a smooth point $[\alpha'_d]$ is $2d-14$, so the dimension of $\mathsf{M}_X(2;-1,d)$ at the bundle $[F'_d]$ corresponding to $[\alpha'_d]$ is also $2d-14$. Thus taking a section of the general bundle F'_d we obtain a curve $(Z)'$ with $\mathrm{h}^1(N_{(Z)',X}) = 0$, so $\mathrm{h}^0(N_{(Z)',X}) = d$. Then the Hilbert scheme $\mathcal{H}_{d,1}(X)$ is d-dimensional and smooth at $[(Z)']$. $\square$

End of the proof of (2.16). Consider a general hyperplane section S_{22} of X. It is a K3 surface of Picard number $\rho(S_{22}) = 1$. Consider then F_d, as defined in Proposition 4.5. Restricting F_d to S_{22} we get a stable rank-2 vector bundle on S_{22}. The moduli space $\mathsf{M}_{S_{22}}(2;-1,d)$ is then smooth and projective of dimension $-\chi(\mathrm{End}(S_{22},F_d))-2$. It is immediate to check that $\dim(M_{S_{22}}(-1,d)) = 4d-28$. Hence $d \geq 7$. $\square$

5. Canonical and Half canonical curves

In this section we will prove the existence of the bundles of Cases (iv) and (v) of Lemma 2.16. Case (v) will be dealt with in Subsection 5.1 while Case (v) is treated in Subsection 5.2.

5.1. **Half–canonical curves.**

We will prove the existence of a smooth half–canonical curve C_{60}^{59} by a deformation argument.

Lemma 5.1. *There exists a smooth curve $Z = C_{60}^{59}$ in X of degree 59 and genus 60, given as the zero locus of a section of an aCM vector bundle $\mathscr{F}_{-1,15}(2)$. We have $\omega_Z \simeq \mathscr{O}_X(2)_{|Z}$. The aCM bundle $\mathscr{F}_{-1,15}$ specializes to the non–aCM bundle F_{15}.*

Proof. Recall by Proposition 4.4 that there exists an elliptic curve $C = C_1^{15}$ such that C is contained in no hyperplane and $\mathrm{h}^1(J_{C,X}) = 1$. The vector bundle F_{15}^* then has a unique section vanishing along C according to Proposition 4.5.

Now by Theorem 4.6 the moduli space $\mathsf{M}_X(2;-1,15)$ is smooth and 16-dimensional at a general $[F_{15}]$. On the other hand, consider the irreducible component of $\mathsf{M}_X(2;-1,15)$ containing $[F_{15}]$ and an an open neighbourhood of $[F_{15}]$ contained in this component. Consider a point $[F'_{15}]$ belonging to this neighbourhood, and represented by a stable bundle F'_{15}, where F'_{15} is not isomorphic to F_{15}.

Now suppose $F'_{15}(1)$ has a nontrivial global section s, and recall that $\mathrm{h}^0(F_{15}) = 0$ by stability. The zero locus of s would then be a curve C' of degree 15 and arithmetic genus 1. Therefore s would give a point $[C']$ in $\mathcal{H}_{15,1}(X)$. The point $[C']$ does not coincide with $[C]$, for otherwise $J_{C',X} \simeq J_{C,X}$ would yield $F'_{15} \simeq F_{15}$.

Being $\mathcal{H}_{15,1}(X)$ smooth of dimension 15 at $[C]$, the above discussion proves that the map $\tau : \mathcal{H}_{15,1}(X) \to \mathsf{M}_X(2;-1,15)$ is an open embedding at $[C]$ and its image is the codimension-1 locus $\{[F'_{15}] \in \mathsf{M}_X(2;-1,15) \,|\, \mathrm{h}^0(F'_{15}(1)) \neq 0\}$. So for general $[F'_{15}]$ we will have $\mathrm{h}^0(F'_{15}(1)) = 0$.

Now since $\chi(F'_{15}(1)) = 0$ we also get $\mathrm{h}^1(F'_{15}(1)) = 0$. Therefore we put $\mathscr{F}_{-1,15} = F'_{15}$ and $\mathscr{F}_{-1,15}$ is aCM. Finally, by Castelnuovo–Mumford regularity $\mathscr{F}_{-1,15}(2)$ is globally generated, so a general section vanishes along a smooth curve Z with the required invariants. $\square$

Remark 5.2. Any aCM stable bundle of type $\mathscr{F}_{-1,15}$ is the cohomology of a monad of type (23) with $d = 15$. Indeed it suffices to apply the proof of Theorem 4.6 to $\mathscr{F}_{-1,15}$.

5.2. **Canonical curves.** Here we will prove the existence of a smooth canonical curve in X by exhibiting the bundle $\mathscr{F}_{0,4}$ of Lemma 2.16.

Lemma 5.3. *Given a general homomorphism $\alpha : U^{\oplus 2} \to (Q^*)^{\oplus 2}$, the sheaf* $\operatorname{coker}(\alpha)$ *is a vector bundle of type $\mathscr{F}_{0,4}$.*

Proof. Define the 2-dimensional vector spaces W_1 and W_2 so that $\alpha : W_1 \otimes U \to W_2 \otimes Q^*$. Let $p_1 : k \to W_1$ (resp., $p_2 : W_2 \to k$) be an element of $\check{\mathbb{P}}(W_1)$ (resp., an element of $\mathbb{P}(W_2)$). To the pair (p_1, p_2) we associate the map $U \to Q^*$ and we get the morphism η_α:

$$\begin{aligned} \eta_\alpha : \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^2 = \mathbb{P}(B) \\ (p_1, p_2) &\mapsto (p_2 \otimes \mathrm{id}_{Q^*}) \circ \alpha \circ (p_1 \otimes \mathrm{id}_{U^*}) \end{aligned}$$

For general α the map η_α is a $2:1$ cover. Suppose now that α is not injective as a bundle map at a given point x of X. Then there exists $p_1 : k \to W_1$ such that, for any $p_2 : W_2 \to k$, the map $\eta_\alpha(p_1, p_2)$ is zero over x. Equivalently x lies in the conic whose ideal is $\operatorname{coker}(\eta_\alpha(p_1, p_2))$. Being η_α a finite map, this means that x lies in the pencil of conics parameterized by $p_2 \in \mathbb{P}(W_2)$, contradicting Lemma 3.2. Therefore $\operatorname{coker}(\alpha)$ is locally free and has the required Chern classes by a straightforward computation.

From the exact sequence:

$$0 \to U^{\oplus 2} \to (Q^*)^{\oplus 2} \to \mathscr{F}_{0,4} \to 0$$

we see immediately that $\mathrm{h}^0(\mathscr{F}_{0,4}) = 0$ and $\mathrm{h}^1(\mathscr{F}_{0,4}(t)) = 0$ for any $t \in \mathbb{Z}$, indeed U and Q^* are aCM bundles.

Therefore $\mathscr{F}_{0,4}$ is stable and aCM, indeed Serre duality gives $\mathrm{h}^2(\mathscr{F}_{0,4}(t)) = \mathrm{h}^1(\mathscr{F}_{0,4}(-1-t)) = 0$ for all $t \in \mathbb{Z}$. Finally, one can compute the following:

$$\begin{aligned} \mathrm{h}^1(Q^* \otimes \mathscr{F}_{0,4}(1)) &= 0 \\ \mathrm{h}^2(U \otimes \mathscr{F}_{0,4}(1)) &= 0 \\ \mathrm{h}^3(E \otimes \mathscr{F}_{0,4}(1)) &= 0 \end{aligned}$$

So by Corollary 2.15 we get that $\mathscr{F}_{0,4}(1)$ is globally generated hence the zero locus of its general global section is the required canonical curve. ☐

Lemma 5.4. *Any aCM stable vector bundle of type $\mathscr{F}_{0,4}$ is the cokernel of a map $\alpha : U^{\oplus 2} \to (Q^*)^{\oplus 2}$.*

Proof. The argument is analogous to that of Theorem 4.6. In this case we find:

$$\begin{aligned} &\mathrm{h}^p(U \otimes \mathscr{F}_{0,4}) = 0 && \text{for } p \neq 1 \\ &\mathrm{h}^p(K \otimes \mathscr{F}_{0,4}) = 0 && \text{for } p \neq 1 \\ &\mathrm{h}^p(E \otimes \mathscr{F}_{0,4}) = 0 && \text{for all } p \end{aligned}$$

We conclude $\mathrm{h}^1(U \otimes \mathscr{F}_{0,4}) = -\chi(U \otimes \mathscr{F}_{0,4}) = 2$ and $\mathrm{h}^1(K \otimes \mathscr{F}_{0,4}) = -\chi(K \otimes \mathscr{F}_{0,4}) = 2$, so the statement follows from Theorem 2.14. ☐

Remark 5.5. Summing up we found that an open subset of a component of $\mathsf{M}_X(2; 0, 4)$ is isomorphic to an open subset of the variety of Kronecker modules

$$\mathbb{P}(W_1^* \otimes W_2 \otimes B) / \mathsf{SL}(W_1) \times \mathsf{SL}(W_2)$$

where W_1 and W_2 are 2-dimensional vector spaces. In particular it is unirational and generically smooth of dimension 5.

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